Analytic solution for the spectral density and localization properties of complex networks

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Large complex systems are ubiquitous in the real world, ranging from physical and biological to social and technological systems [1]. The adjacency random matrix plays a central role in this context, as it describes the interactions among the individual elements that compound these large complex systems. The empirical spectral density of the adjacency random matrix and the localization of its eigenvectors are key quantities to understand a variety of dynamical processes on complex networks (or random graphs) [2, 3]. The spectral and localization properties of the adjacency random matrix are given by functions of the diagonal elements G_{ii} of the resolvent matrix $\mathbf{G}$. The imaginary part of G_{ii} determines the local density of states (LDOS), which counts the number of states at a certain eigenvalue λ at node i. The average of the LDOS over all the nodes determines the empirical spectral density, while The average of $|G_{ii}|^2$ gives information about eigenvector localization throughout the inverse participation ratio (IPR), which characterizes the volume of the eigenvectors. The probability density function of G_{ii} satisfies a system of distributional equations [4, 5, 6], providing a solid foundation to study the spectral and localization properties of *heterogeneous* random graphs. Heterogeneity is broadly associated with local fluctuations in the graph structure, such as randomness in the degrees or in the interaction strengths between the nodes (the degree of a given node counts the number of edges attached to it). Although the resolvent distributional equations have led to enormous progress, they admit analytical solutions only for random graphs with a homogeneous structure [5, 7, 8]. In a recent paper [9], the resolvent equations for the configuration model of random graphs with a geometric degree distribution have been studied in the high connectivity limit, i.e., when the average degree c becomes infinitely large. It is shown in this paper that the average resolvent satisfies a transcendental equation, and the spectral density diverges at the center of the spectrum. These findings are interesting because they suggest the existence of a new class of solutions for the distributional equations of the resolvent in the high connectivity regime, which lies between the sparse (when the average connectivity is finite) and the dense regime (when the random graph becomes fully connected). Moreover, these analytical results also imply that the spectral density of random graphs in the high connectivity limit is not typically governed by the Wigner semicircle law of random matrix theory [10], as it is rigorously proven in [11]. Indeed, the Wigner law universality only holds for random graphs with degree distributions that become highly concentrated around its mean value for $c \to \infty$. In other words, the average connectivity is large, but the fluctuations in the network are still relevant for the spectral properties. The analytical results obtained in [9] are limited, however, to a geometric degree distribution. In this work [12], we generalize the results of [9] and derive analytical solutions for the resolvent distributional equations of random graphs with arbitrary degree distributions in the high-connectivity limit. In this context, we perform a detailed analysis of the impact of degree fluctuations on the spectral density, the inverse participation ratio, and the distribution of the local density of states. For random graphs with a negative binomial degree distribution, we show that all eigenvectors are extended and that the spectral density unveils a logarithmic or a power-law divergence when the variance of the degree distribution is sufficiently large. We elucidate this singular behaviour by showing that the distribution of the LDOS at the center of the spectrum exhibits a power-law tail determined by the variance of the degree distribution. In addition, we show that in the regime of weak degree fluctuations the spectral density of random graphs with a negative binomial degree distribution has finite support, which promotes the stability of large complex systems on random graphs.

We consider a simple and undirected random graph with N nodes. The network topology is specified by the components of the adjacency random matrix $\mathbf{A}$. We generate $\mathbf{A}$ according to the configuration model of networks [1, 13, 14] in which a random graph is chosen uniformly at random from the set of all random graphs with a given degree sequence $K_1, \dots, K_N$ generated from a prescribed degree distribution p_k. In the high connectivity limit, the spectral density and the inverse participation ratio are, respectively, given by

$$\rho_\epsilon(\lambda) = \frac{1}{\pi}\mathrm{Im}\left[\int_0^\infty d\kappa \frac{\nu(\kappa)}{z - \kappa J_1^2 \langle G\rangle}\right], \tag{1}$$

$$\mathcal{I}_\epsilon(\lambda) = \frac{\epsilon}{\pi \rho_\epsilon(\lambda)} \int_0^\infty d\kappa \frac{\nu(\kappa)}{|z - \kappa J_1^2 \langle G\rangle|^2}, \tag{2}$$

with $z = \lambda - i\epsilon$ lying on the lower complex half-plane and J_1^2 denoting the variance of the distribution that defines the coupling strengths between the graph nodes. The variable $\langle G\rangle$ satisfies the fixed-point equation

$$\langle G\rangle = \int_0^\infty d\kappa \frac{\nu(\kappa)\,\kappa}{z - \kappa J_1^2 \langle G\rangle}. \tag{3}$$

In the high connectivity limit, the fluctuations in the random graph are captured by the empirical distribution of the re-scaled degrees $\nu(\kappa)$, which is defined as

$$\nu(\kappa) = \lim_{c\to\infty} \sum_{k=0}^{\infty} p_k \delta\left(\kappa - \frac{k}{c}\right). \tag{4}$$

The solution of the self-consistent equation given by (3) determines each and every equation of this work.

For a negative binomial degree distribution $p_k^{(b)}$, one can investigate the role of degree fluctuations on the spectral and localization properties of random graphs in terms of a single parameter in the high connectivity limit, given by the relative variance of $p_k^{(b)}$, i.e.

$$\frac{1}{\alpha} = \lim_{c\to\infty} \frac{\sigma_b^2}{c^2}. \tag{5}$$

By considering the negative binomial distribution, we obtain a simple expression for the distribution of the LDOS at $z = 0$, viz.

$$P_0(y) = \frac{\alpha^\alpha}{\Gamma(\alpha) J_1^\alpha} \frac{e^{-\frac{\alpha}{J_1 y}}}{y^{\alpha+1}}. \tag{6}$$

Equation (6) reveals the unbounded character of the LDOS fluctuations at $\lambda = 0$. We show in this work that the spectral density diverges for $\alpha \leq 1$ at $\lambda = 0$ (see Figure 1). In this regime, the above result helps us to clarify this singular behaviour. The power-law tail of (6) exhibits a divergence in the q-th moment $\overline{y^q} = \int_0^\infty dy y^q P_0(y)$ for $\alpha \leq q$, which explains the singularity of the spectral density at $\lambda = 0$, for $\alpha \leq 1$. In addition, the non-singular, $\epsilon-$independent behaviour of (6), confirms the extended phase of the eigenvectors in the high connectivity limit at $\lambda = 0$.

In summary, this work unveils non-trivial results for the resolvent distributional equations of undirected random graphs with a heterogeneous structure in the high connectivity limit. All of our results are determined solely in terms of the empirical spectral density of the re-scaled degrees and the complex variable $\langle G\rangle$, in which the latter satisfies a self-consistent equation.

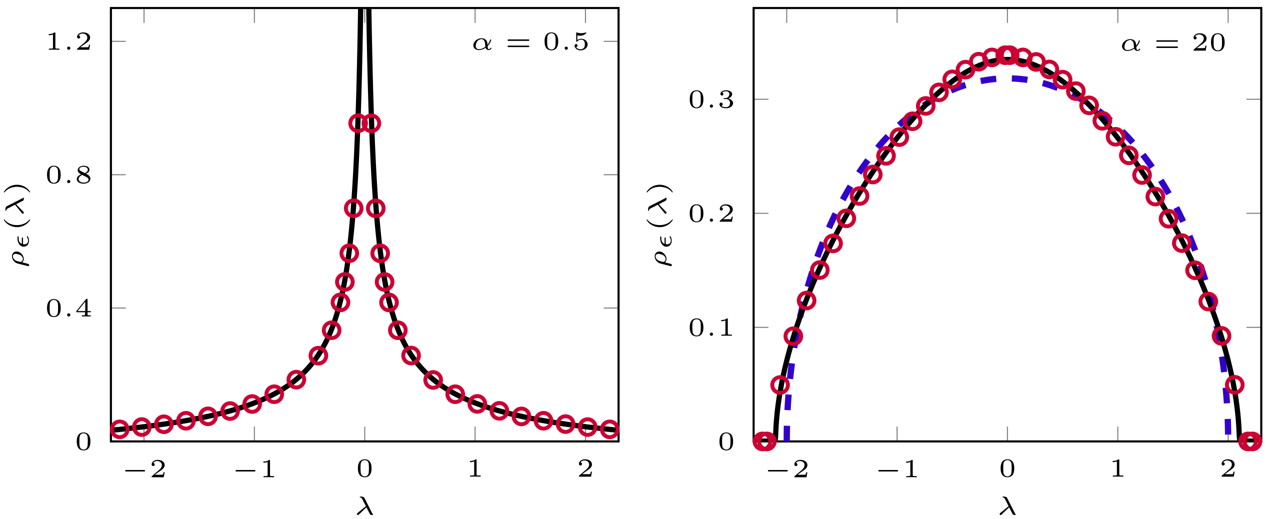

Figure 1: The spectral density of random graphs with a negative binomial degree distribution in the high-connectivity limit. The parameter $1/\alpha$ controls the relative variance of the degree distribution (see Eq. (5)). The solid lines are the theoretical results derived from solving Eqs. (1) and (3) for $\epsilon = 10^{-3}$ and $J_1 = 1$. The red circles are numerical diagonalization results obtained from an ensemble of $10^4 \times 10^4$ adjacency random matrices. The dashed blue curve in the right panel represents the Wigner semicircle law.

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